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Spinning modes on axisymmetric jets. Part 1

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Linear instability analysis is applied to the slowly diverging mean profile of a turbulent axisymmetric jet and used to predict the transverse structure and axial evolution of large-scale wavelike modes with azimuthal wavenumber $m = 1$. Comparisons are made with measurements of filtered velocity fluctuations, and of pressure fluctuations, taken in a jet with coherent forcing at the exit plane, at Strouhal numbers $St = fD/U_0$ around 0.5, $f = \omega/2\pi$ being the frequency, D the nozzle diameter and U_0 the mean centreline exit velocity. The transverse structure at each axial station is well predicted by linear theory, as is the phase speed and its variation with axial distance. The downstream evolution of amplitude is much less well predicted, presumably because of cumulative nonlinear effects in the experiments, though the inclusion of mean-flow divergence itself constitutes a significant improvement over the theory for parallel flow, and in some cases permits calculation of the wave evolution well into the decay phase without any reference to viscous effects on the disturbance.

1. Introduction

This paper aims to make a contribution to the modelling of orderly large-scale structures in high-Reynolds-number axisymmetric turbulent jets. Attention has, for the most part, been confined to axisymmetric modes, both in experimental work (Crow & Champagne 1971; Chan 1974; Moore 1977; and many others) and in theoretical (Michalke 1971; Grant 1974; Crighton & Gaster 1976; Acton 1980). That emphasis is entirely natural for reasons of both experimental and theoretical convenience. First, if one is using controlled forcing to raise the large-scale structures above the random background, then low-frequency acoustic forcing by plane waves generated by a loudspeaker in a jet-rig plenum chamber provides a simple arrangement for exciting axisymmetric jet modes, whereas modes with azimuthal variation can only be generated individually by a much more complicated loudspeaker system. Secondly, if one attempts to model the structures with concentrations of vorticity (thereby allowing a nonlinear treatment of their dynamics), one can much more readily handle a train of axisymmetric ring vortices (Grant 1974; Morfey 1979; Acton 1980; among many) than the helical vortex filaments which would be needed to describe modes with azimuthal variation. It is equally clear, however, that structures with low-order azimuthal variation must also be taken into account. Such structures are often see in flow-visualization studies when there is no systematic plane-wave forcing and can presumably be as easily excited as axisymmetric structures in a real aeroengine exhaust flow. Moreover, all instability analyses indicate that the growth rates of first-order azimuthal modes are at least comparable with those of axisymmetric modes (and indeed, in one particular case, that of the bell-shaped profile adopted by a jet far beyond the end of the potential core, first-order azimuthal modes

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may continue to grow while axisymmetric modes can only decay (Batchelor & Gill 1962)).

Experiments on the controlled forcing, with azimuthal variation, of high-Reynolds-number jets have been reported by Chan (1977) and by Bechert & Pfizenmaier (1977). The intent of the latter authors was to investigate whether the broadband amplification phenomenon would occur with azimuthally varying modes to an extent similar to that found in the original plane-wave experiments of Bechert & Pfizenmaier (1975) and Moore (1977); they found, indeed, very similar results, except that the forcing tone, which was the dominant feature of the far-field sound in the plane-wave case, was dramatically reduced (by some 30 dB) in the $m = 1$ and $m = 2$ azimuthal mode cases, the tone in those cases being acoustically 'cut off' in the jet pipe but still strong enough to excite large-scale orderly structures on the jet shear layers. Bechert & Pfizenmaier (1977) did not, however, report any results for the jet structure, and confined their presentation to the acoustic far field.

Experiments with spinning modes $m = 1, 2$ were reported by Chan in 1977. While we have attempted to compare our own theoretical results with Chan's data, we do not feel that the attempt constitutes a useful test of our calculations. For first, Chan's (limited) data involve pressure fluctuations only, which inevitably vary slowly in the axial and radial directions, and we attach far greater importance to the results for velocity fluctuations, which show much more structure. The thesis of Strange (1981), upon which this paper and Part 2 are based, is currently the only source of data involving velocity fluctuations. Secondly, in his presentation of data on wavelengths and phase velocities deduced from pressure measurements, Chan (1977, figure 3) does not indicate the axial locations at which the data were taken (and there is considerable variation with x; see figure 10 below). Thirdly, although Chan does give data for the evolution, with axial distance, of the pressure fluctuations in the shear layer and on the centreline, we do not consider prediction of those variations to be a significant test in itself, for in Crighton & Gaster (1976) it was shown that linear theory might reasonably predict the pressure amplification downstream yet fail to predict, to any comparable accuracy, the much larger amplification that occurs in the velocity fluctuations. On the theoretical side, no helical vortex-filament modelling (or indeed any kind of vortex-filament modelling involving an axial component of vorticity) has yet been attempted for non-axisymmetric modes to complement the ring-vortex description of axisymmetric modes. However, many authors have given results for non-axisymmetric modes in a modelling of these modes as spinning instability waves amplifying and decaying on the mean profile, with account taken in varying degrees both of nonlinearity and mean flow divergence. Weakly nonlinear disturbances to a slowly diverging shear layer continue to present serious conceptual problems (Huerre 1980; Huerre & Scott 1980) and the most successful treatments of nonlinearity have used a pragmatic mixture of energy equations, shape assumptions and closure hypotheses (for the most comprehensive discussion see Mankbadi & Liu 1981). Here we deal only with the linear instability problem for spinning modes, but deal rationally with the effect of mean-flow divergence in the way carried out for axisymmetric modes by Crighton & Gaster (1976). Plaschko (1979) has carried out a calculation very similar to the present one, though he did not (as we do, in §2) justify the approach, nor was he able to calculate the transverse mode shapes with complete success. Further, the only data available for comparison with his results were then those of Chan (1977), with deficiencies mentioned earlier. Thus Plaschko was able only to make qualitative comparison of his theory with experiment, the comparison (which we in no way dispute) relating to the selection of a different 'preferred' Strouhal number for different mode numbers m, and different variations with axial distance of the phase speed of the pressure signal for different values of m.

There is thus a case for the reporting of further experimental and theoretical work on the problem of non-axisymmetric structures and their modelling in terms of spinning instability wave modes. This paper, and its sequel, Part 2, give a selection of data and calculations from the large collection in Strange (1981). In Part 1 we present results of a now familiar instability calculation for spinning modes of wave-number $m = 1$ on a slowly diverging mean turbulent profile (with a good fit of an analytical form to our own measurements) and compare them with hot-wire measurements of filtered velocity fluctuations in the flow, and with pressure-fluctuation measurements in the near field. Theory and experiment are confined to axial locations upstream of the end of the potential core, because the aim of this study – and of most previous studies of forced-jet behaviour – is an improved understanding of noise-production mechanisms. It is generally believed that much of the noise production occurs in the early part of the jet (except in the case of high-speed flow), and indeed Moore (1978) has shown that the noise sources *for all frequencies in a forced jet* are concentrated in a spatial region between two and four diameters downstream of the nozzle exit.

In our experiments a controlled forcing, with the required azimuthal variation and with Strouhal number around 0.5, was imposed at the nozzle exit plane. In the sense that the forcing was strong enough to increase the broadband turbulence levels and far-field sound by several decibels (for plane-wave excitation this would require a velocity fluctuation, uniform over the exit plane, in excess of about 0.1 % of the mean velocity, according to Moore 1977) the response of the instability wave might be classified as *significantly nonlinear*, and, in the axisymmetric case, high-speed films clearly show shear-layer break-up into vorticity concentrations which then pair and merge as they travel downstream. On the other hand, spectra of the velocity and pressure fluctuations show that the behaviour of the forcing tone itself is only *weakly nonlinear* in the sense that, at any location, integral harmonics of the forcing can be seen, but these are weak compared with the fundamental, and decrease rapidly in level from one harmonic to the next. One might therefore expect a reasonable *local* prediction of the large-scale instability mode structure, but suspect that nonlinear effects would accumulate with axial range and invalidate the prediction by linear theory of the cumulative growth experienced by the wave. This is indeed a consistent interpretation of the results of §3.

Part 2 will confine itself to the experimental programme, dealing with the dramatic and varied distortion of the mean jet flow that can be achieved by relatively low-level forcing in modes with $m = 0$, 1 and 2. It was shown by Crow & Champagne (1971) that plane-wave forcing could significantly distort the mean profile over the first six diameters or so, the effect on the far-downstream jet being equivalent to an upstream shift of the virtual origin by some two diameters. We shall show in Part 2 that similar levels of forcing in the $m = 1$, 2 modes can lead to a more severe distortion of the mean flow over the first *twelve* diameters. Part 2 will also deal with the broadband response of the jet to forcing, both in the flow field itself and in the acoustic far field.

2. Linear instability analysis

Since the style of analysis to be used here has already been used for axisymmetric modes by Crighton & Gaster (1976) and for precisely the present spinning-mode problem by Plaschko (1979), only the briefest summary will be given. First, however, the *basis* for the calculation should be noted, because it is not obvious that one can conduct linear inviscid instability analysis on a diverging mean turbulent jet profile which does not itself satisfy the Euler equations.

Suppose that L is a shear-layer thickness, D the jet diameter, and $\epsilon = L/D \ll 1$. Suppose, further, that the shear-layer fine-scale turbulence characteristics scale on L, U_0, while the large-scale instability wave scales on D, U_0, where U_0 is the mean centreline exit velocity. Decompose the flow variables into an ensemble mean, plus a random turbulent field (denoted by $'$), plus a large-scale field (denoted by $\sim$), so that

$$\boldsymbol{u} = \boldsymbol{U}\left(\frac{\boldsymbol{x}}{L}\right) + \boldsymbol{u}'\left(\frac{\boldsymbol{x}}{L}, \frac{U_0 t}{L}\right) + \tilde{\boldsymbol{u}}\left(\frac{\boldsymbol{x}}{D}, \frac{U_0 t}{D}\right), \tag{2.1}$$

and correspondingly for the pressure. Then, perform a time-average of the Navier–Stokes equations, over a time T such that $T \gg L/U_0, D/U_0$, subtract the time-averaged equations from the full Navier–Stokes equations and perform a second average, this time such that $L/U_0 \ll T \ll D/U_0$. In the resulting equation, neglect (i) linear viscous terms $\nu\nabla^2\tilde{\boldsymbol{u}}$, these normally being insignificant in the amplification stage and masked, in the present problem, in the decay stage by other effects (cf. §4), (ii) non-linear terms $\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}}$, and (iii) linear terms $\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}'$ representing the interaction between the instability wave and fine-scale background turbulence (an interaction which might, if required, be handled using Crow's (1968) theory of the viscoelastic response of turbulence to large-scale weak fields). Then the equations for $\tilde{\boldsymbol{u}}$ (for *incompressible* fluctuations) reduce to those for a linear perturbation to a hypothetical flow $\boldsymbol{U}(\boldsymbol{x}/L)$, namely

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)\tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{U} = -\nabla\tilde{p}, \\ \operatorname{div}\tilde{\boldsymbol{u}} = 0. \end{aligned}\right\} \tag{2.2}$$

We can now refer the reader to Crighton & Gaster (1976), Plaschko (1979) and Strange (1981) for details of the way in which a multiple-scales expansion (for $\epsilon \ll 1$) is used to produce, for each real frequency ω and mode number m, a solution, to leading order, for $\tilde{\boldsymbol{u}}$ and $\tilde{p}$ which is uniformly valid in the axial and radial coordinates (x, r) of a cylindrical polar system (x, r, θ). It emerges that

$$\tilde{p}(x,r,\theta,t) = A(x)\,\phi_p(r,x)\exp\left\{\mathrm{i}\int_{x_0}^{x}\alpha(x)\,\mathrm{d}x - \mathrm{i}\omega t + \mathrm{i}m\theta\right\}, \tag{2.3}$$

and correspondingly for the velocity components $(\tilde{u}_x, \tilde{u}_r, \tilde{u}_\theta)$ with the replacement of ϕ_p by $(\phi_x, \phi_r, \phi_\theta)$. Here $\alpha(x)$ is the slowly changing instability wavenumber for the local parallel flow at station x, for which the eigenvector has components $(\phi_x, \phi_r, \phi_\theta, \phi_p)$ governing the transverse distributions of $(\tilde{u}_x, \tilde{u}_r, \tilde{u}_\theta, \tilde{p})$, with x appearing only parametrically. The frequency ω and integral azimuthal wavenumber m are prescribed *real* quantities. The amplitude equation (a misnomer, because α and $\boldsymbol{\phi}$ are complex and vary with x) is of the form

$$\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{n(x)}{m(x)}A = 0, \tag{2.4}$$

where $n(x)$ and $m(x)$ are certain functionals of $\boldsymbol{U}$, obtained from radial integrations involving $\boldsymbol{U}$ and its derivatives along with $\boldsymbol{\phi}$, its derivatives, and its adjoint $\hat{\boldsymbol{\phi}}$. Integration of (2.4) yields

$$\tilde{p} = A_0\,\phi_p(r,x)\exp\left\{\mathrm{i}\int_{x_0}^{x}\alpha(x)\,\mathrm{d}x - \int_{x_0}^{x}\frac{n(x)}{m(x)}\mathrm{d}x - \mathrm{i}\omega t + \mathrm{i}m\theta\right\}, \tag{2.5}$$

and similar expressions for the components of $\tilde{\boldsymbol{u}}$. These expressions are *independent* of the particular normalization adopted for $\boldsymbol{\phi}$.

In §3, numerical calculations of the quantities $\tilde{p}, \tilde{\boldsymbol{u}}$ will be presented for a specific choice of mean profile $U(r, x)$, and compared with experimental data. In relation to that comparison, two points need to be made. First, in the experiments we are dealing with modes that are phase-locked by coherent exit-plane forcing, and the large-scale component of (2.1) can be unambiguously determined by averaging over a period at all stations up to, say, $x/D = 6$, for which the phase locking is essentially complete. Secondly, in the theory the interaction with the jet tailpipe has been completely ignored. This interaction has been examined for a plane splitter plate by Orszag & Crow (1970) and for the circular pipe by Rienstra (1983), with the general conclusion that the interaction becomes insignificant compared with the growing instability wave at downstream distances greater than about half an instability wavelength. For the plane-wave modes studied by Crow & Champagne (1971) the preferred mode wavelength is $2.38D$, so that we conclude that the effect of the tailpipe is small for $x \gtrsim D$ and accordingly begin our integrations for the axial evolution at $x = D$. We believe that our analysis is still valid for $x > D$, despite the fact that the shear-layer thickness is not very small compared with D. The essential requirement is that the turbulence be fine-grained on the scale of the instability wavelength, and the latter is typically $2D$ while the eddy integral scale is perhaps one-third of the local shear-layer thickness.

3. Numerical results and comparisons with experiment

For the purpose of numerical calculations, the eigenvalue problem for locally parallel flow was formulated in terms of a single dependent variable, the pressure fluctuation. At each axial station the eigenvalues were determined by an iterative procedure, using a finite-difference method to integrate in the radial direction. Radial integrations were performed both outwards, from near the jet axis, and inwards, from $r = 2.5D$, and the resulting pressures and their gradients were matched at an intermediate radius. In this manner the eigenvalues, typically $O(1)$, were obtained to an accuracy of at least four significant figures for a step length of $0.0033R$, where $R = \frac{1}{2}D$ is the jet exit radius, and the full transverse eigensolution structure could then be evaluated, locally, at each axial station. Plaschko (private communication in relation to his 1979 paper) used a much simpler scheme, involving either outward integration from a point near the axis $r = 0$ and matching to an asymptotic solution at a large r, or the reverse. Such a scheme obviously has difficulties associated with the fact that the asymptotic solution for large r (involving the Bessel function $K_m(\alpha r)$) is singular at $r = 0$, while that for small r, involving $I_m(\alpha r)$, is singular as $r \rightarrow \infty$, and as a result Plaschko was unable to determine satisfactorily the large-r behaviour of the eigenfunctions, although his determination of the instability wavenumbers α appears to have been accurate in so far as he was able to reproduce the downstream evolution of an axisymmetric mode precisely as calculated earlier by Crighton and Gaster. The finite-difference outward and inward integration scheme used in the present study suffers from no such disadvantages; and as a preliminary check we also used it, in the axisymmetric case $m = 0$, to confirm the results of Crighton & Gaster, and in the case $m = 1$ to confirm the results published by Plaschko.

Axial integrations for (2.5) were started at $x = 1D$, since for $x \lesssim 1D$ it is quite unrealistic to model the jet as doubly infinite, particularly at the Strouhal numbers of interest here where the instability wavelengths are comparable with the jet

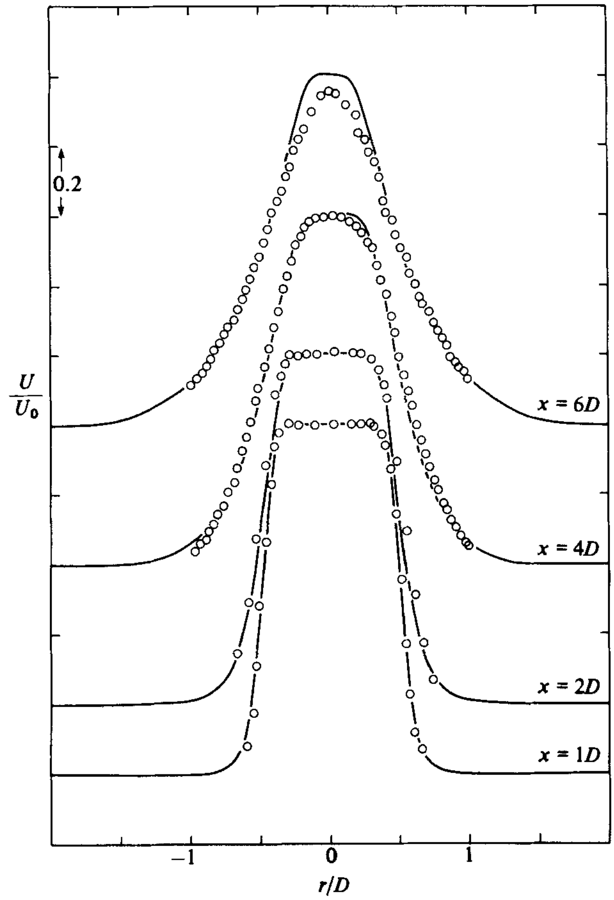

FIGURE 1. Mean-velocity profiles for unexcited jet at axial distances of $1D$, $2D$, $4D$, $6D$; ○, experiment, jet Mach number 0.3; ——, analytical form (3.1).

diameter D. Derivatives in the axial direction were evaluated numerically, using a step length of $0.05D$.

The measurements which appear in the following comparison were obtained as part of the model jet experimental programme to be described in Part 2 of this paper and much more fully in Strange (1981). In that programme, plane-wave and higher-order modes of acoustic excitation were imposed just upstream of the nozzle exit plane of a high-Reynolds-number flow ($10^5 \lesssim Re_D \lesssim 10^6$). The forcing level was sufficiently high that it produced an increase of approximately 6 dB in the far-field broadband noise for all modes of excitation, the Strouhal number of the excitation being close to 0.5.

Before any comparisons are made, attention is drawn to the importance of the mean-flow structure. Michalke (1971), and later Mattingly & Chang (1974), observed that the calculated growth of an instability wave (even of long wavelength) on the first few diameters of the jet is very sensitive to the mean velocity profile used in the calculation and, in particular, that the profile strongly affects the selection of a 'preferred mode'. Indeed, in overlooking this feature Crow & Champagne (1971) rejected even the possibility that the development of large-scale structures on a circular jet could be modelled by linear spatial instability theory. Subsequently an analytical form

$$U(r) = \frac{U_0}{2}\left\{1 + \tanh\left[b\left(\frac{R}{r} - \frac{r}{R}\right)\right]\right\},$$

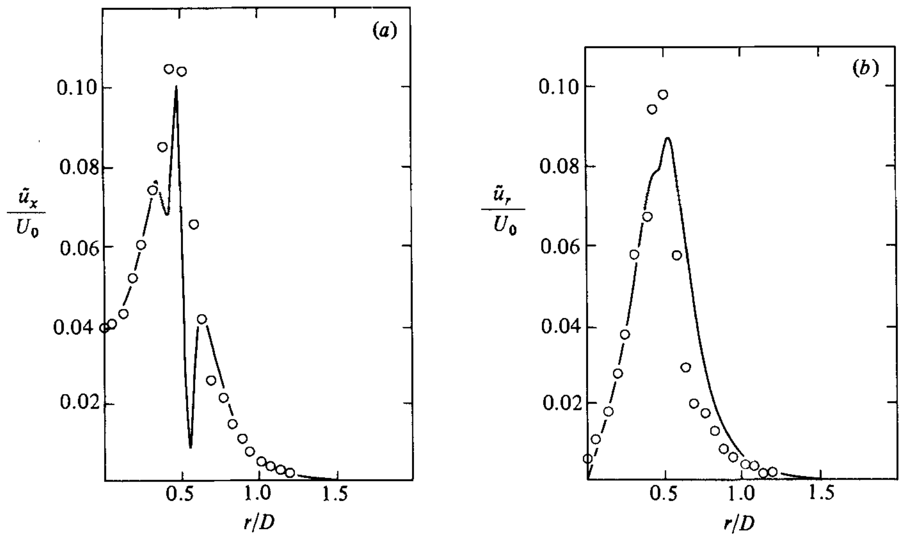

FIGURE 2. Radial distribution of filtered velocity fluctuations at $x = 1D$; (*a*) axial component; (*b*) radial component. Mode index $m = 0$. ○, experiment; $St = 0.544$; ——, theory, same St.

for the mean profile was devised by Michalke (1971) to represent closely, with a particular choice of the constant b, the measured profile of Crow & Champagne at $x = 2D$ (the location where most of their results were taken) and found by him to yield phase speeds and amplification rates which compared favourably with their observations. On the basis of the similarity rules which apply on the initial region ($x \lesssim 6D$) of a turbulent jet (that velocities are invariant with x while lengthscales increase linearly with x), Crighton & Gaster (1976) generalized this form to produce a slowly diverging profile

$$U(r,x) = \frac{U_0}{2}\left\{1+\tanh\left[\frac{25R}{3x+4R}\left(\frac{R}{r}-\frac{r}{R}\right)\right]\right\}, \tag{3.1}$$

and it is this profile which is adopted in the present calculations. The choice is well justified by the comparison with our own experimental profiles in figure 1, where excellent agreement is shown over the first four diameters of the jet.

We first consider the transverse structure of the eigenfunctions. Radial hot-wire traverses were performed at axial stations 1, 2 and 4 diameters downstream of the nozzle exit plane and both axial and radial filtered velocity components were obtained. Figure 2 shows the results at $x = 1D$ for the $m = 0$ mode. The corresponding theoretical results (for incompressible flow and for precisely the same Strouhal number) have been normalized so that the amplitude of the axial velocity component agrees with experiment on the jet centreline (figure 2*a*). Then the induced radial velocity profile, shown in figure 2(*b*), is obtained without any further normalization. In the case of the $m = 1$ (spinning) mode, the predicted radial distribution has been normalized, as in figure 3, by matching the *radial* velocity with experiment on the centreline. Repeating these normalization processes at $x = 2D$ produces the results shown in figures 4 and 5 for the $m = 0$, $m = 1$ modes respectively.

The measurements in the last two figures have been taken in a region of the jet where the schlieren photographs to be shown in Part 2 of this study (and in Strange 1981) indicate that the axial growth of the large-scale wavelike structures is strongly nonlinear. When the jet is subjected to forcing in the *axisymmetric* mode, the

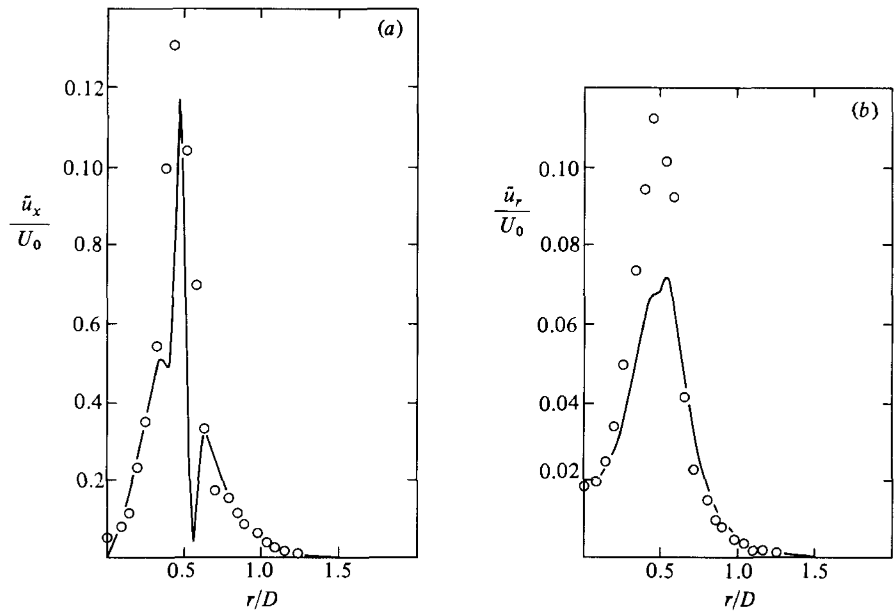

FIGURE 3. As in figure 2, except that now $m = 1$.

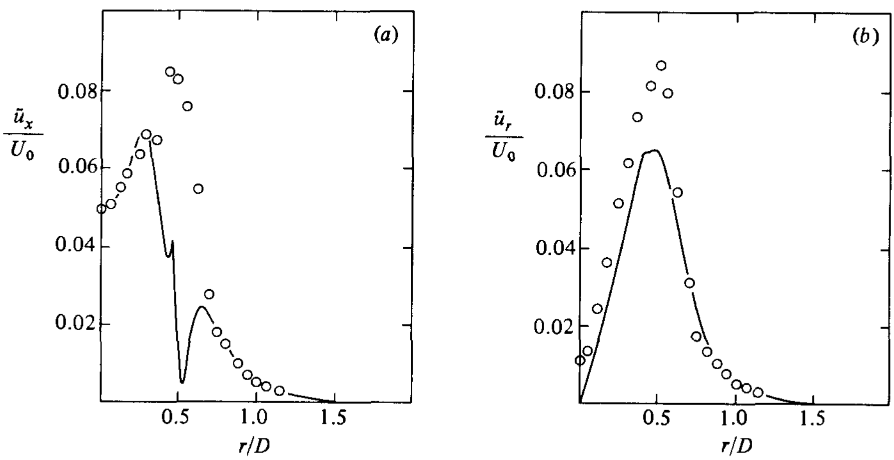

FIGURE 4. Radial distribution of filtered velocity fluctuations at $x = 2D$; (*a*) axial component; (*b*) radial component. Mode index $m = 0$. ○, experiment, $St = 0.544$; ——, theory (renormalized as described in text), same St.

instability wave appears to roll up into a vortex on the shear layer at about $x = 2D$, and, in the next diameter or so downstream, vortex-pairing processes can be observed (as described e.g. by Moore (1977) and very many others). It therefore comes as a somewhat unexpected result that the radial profiles of the axial and radial velocity components are so well approximated by linear theory. On the other hand, in the $m = 1$ mode no such vortex interaction processes can be observed, and the flow is not so obviously nonlinear, although virtually the same level and angular distribution of far-field broadband noise increase is observed in the $m = 0$ and $m = 1$ modes (Bechert & Pfizenmaier 1977; and Part 2 of this paper). To the authors this indicates that an explanation of the broadband amplification is not to be sought in any

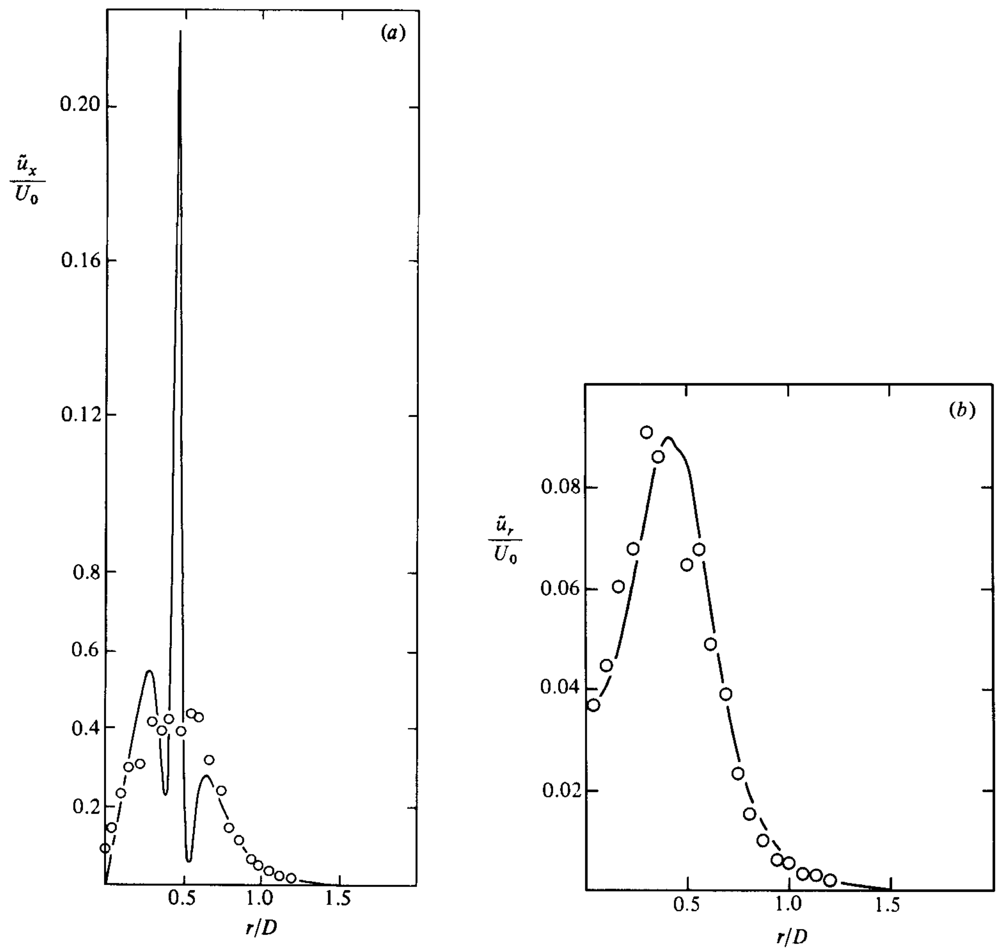

FIGURE 5. As in figure 4, except that now $m = 1$.

mechanism as specific as that of axisymmetric ring-vortex pairing (Ffowcs Williams & Kempton 1978), but rather in the idea that the large-scale wave produces a strong distortion of the mean profile, precipitating the production of vigorous local patches of turbulence. This notion would apply to all large-scale modes provided only that their azimuthal wavenumbers were not large.

We return, in §4, to the good prediction of transverse structure by linear theory.

The radial distribution of the filtered pressure fluctuation is shown in figure 6, where the predictions have been normalized on the peak values in the shear layer (at $r/R = 1$). The experimental results here were taken with the jet running at Mach number 0.5, a condition at which the mean velocity was not measured, but the results are encouraging nonetheless, the more so when one recalls again that the calculation is for incompressible flow. Observe, however, that the radial variations in the velocity fluctuations exhibit far more structure than is displayed by the variations of pressure.

Turning to the axial development of the instabilities, it is, despite the reasonable prediction of transverse structure by linear theory, unrealistic to expect quantitative agreement between linear theory and the observed and apparently nonlinear downstream evolution. The gain in the pressure fluctuations in the $m = -1$ mode (figure 7, where measurements obtained at two different forcing levels are shown)

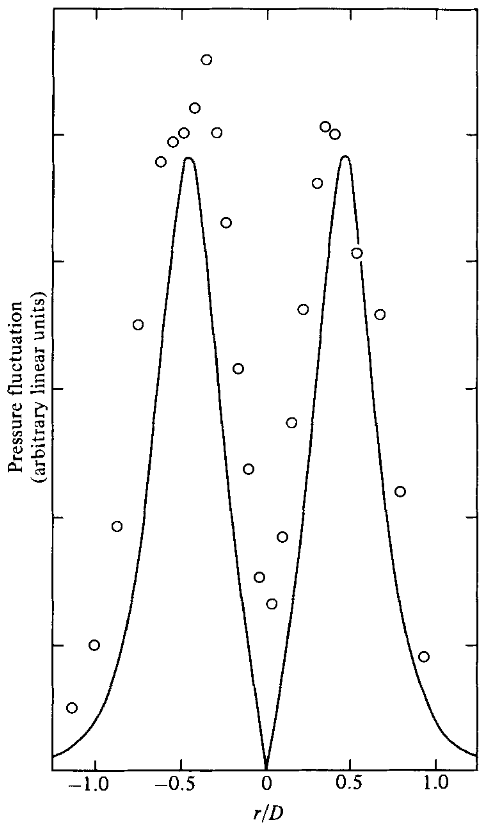

FIGURE 6. Radial distribution of pressure fluctuations at $x = 2D$. Mode index $m = -1$. ○, experiment, $St = 0.555$; ——, theory, same St. The pressure fluctuations are not normalized, and the figure is intended to show only the *shape* of the fluctuations.

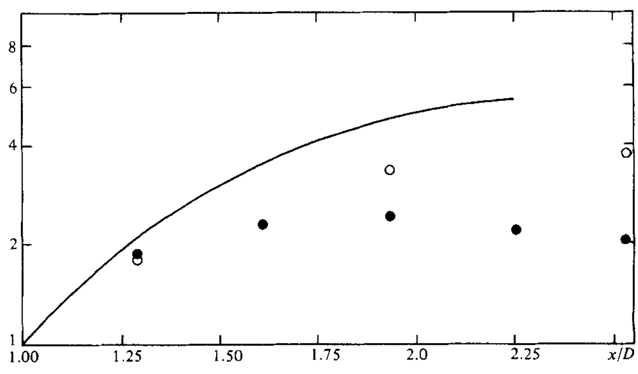

FIGURE 7. Gain in shear-layer pressure fluctuations at $r = R$. Mode index $m = -1$. ○, experiment, $St = 0.555$, low drive level; ●, experiment, $St = 0.555$, high drive level; ——, theory, same St. Jet Mach number for these experimental points is 0.5.

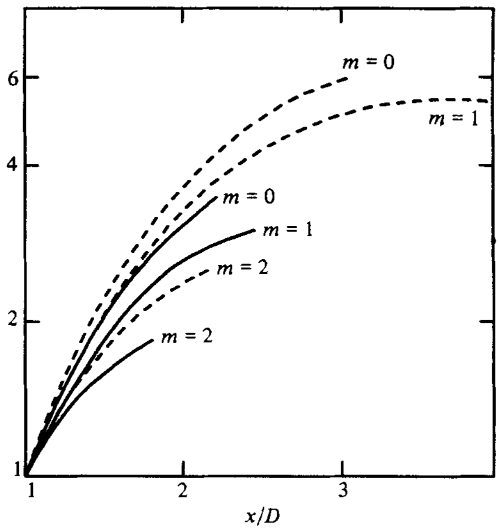

FIGURE 8. Gain in shear-layer pressure fluctuations ($r = R$) at $x/D = 1$: ----, theory; $St = 0.4$; ——, theory, $St = 0.5$. Mode index $m = 0, 1, 2$, as indicated.

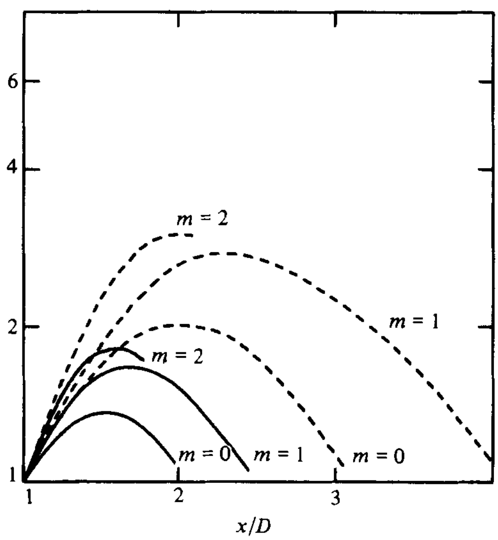

FIGURE 9. Gain in shear-layer axial velocity fluctuation ($r = R$) at $x/D = 1$: -----, theory, $St = 0.4$; ——, theory, $St = 0.5$. Mode index $m = 0, 1, 2$, as indicated.

confirms this expectation. Figure 7 also shows, however, that any simple exponential behaviour of the parallel-flow-theory kind would indicate gains far in excess of those predicted by the analysis for slowly diverging flow (which does not preclude large changes but requires only that – regardless of the wavelength – flow-divergence effects should be *locally* small). Figures 8 and 9, showing the predicted gains in the shear-layer pressure and axial velocity fluctuations at Strouhal numbers of 0.4 and 0.5, serve to emphasize this point. It can be observed here that in some cases the maximum gain condition cannot be achieved by the computation (depending sometimes on the value of m), while in others the calculation can be followed right

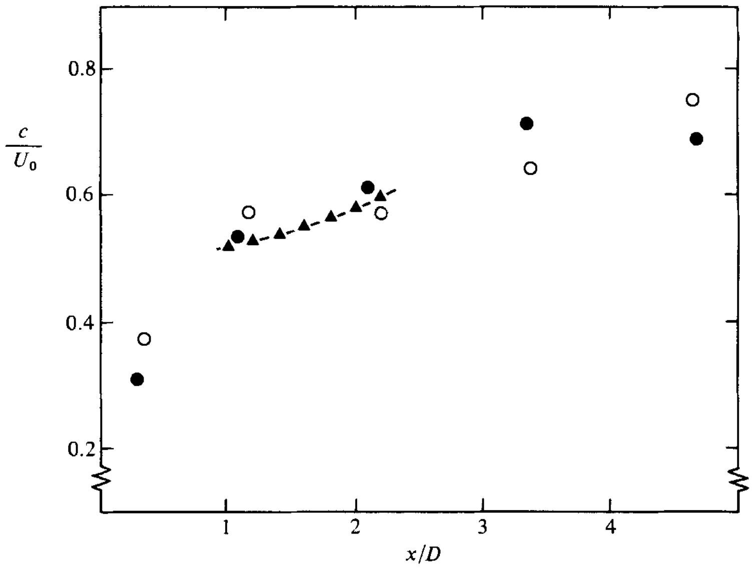

FIGURE 10. Phase speed of shear-layer pressure fluctuation ($r = R$). Mode index $m = -1$. ○, experiment, $St = 0.555$, low drive level; ●, experiment, $St = 0.555$, high drive level; —▲—, theory, equation (3.2).

through until the disturbance has almost decayed back to its original amplitude. However, in *all* cases the calculations shown are restricted to values of x/D for which the local profile remains *unstable* to a spatial instability at the Strouhal number considered. So the 'decay' is brought about entirely by the $n(x)/m(x)$ term in the amplitude equation (2.4) – a term whose presence is excluded by 'slice' analyses which merely replace parallel-flow solutions

$$A_0\,\phi(r)\exp(\mathrm{i}\alpha x - \mathrm{i}\omega t)$$

by

$$A_0\,\phi(r,x)\exp\left(\mathrm{i}\int^x \alpha(x)\,\mathrm{d}x - \mathrm{i}\omega t\right).$$

Observe also (cf. Crighton & Gaster 1976) that the phase speed

$$c(\omega, x, r|\psi) = \omega\Big/\mathrm{Re}\left\{-\mathrm{i}\frac{\partial}{\partial x}\ln\psi(x, r|\omega)\right\} \tag{3.2}$$

of any flow variable ψ can also exhibit an $O(1)$ departure from its parallel flow counterpart. Phase measurements taken for the pressure fluctuation in the $m = -1$ mode were used to infer phase speeds for the pressure, and these are presented – in gratifying agreement with calculation – in figure 10. Again, such agreement cannot be obtained if the local 'slice' analysis is used to define the phase speed simply as $c = \omega/\mathrm{Re}\,\alpha(x)$, a definition that would, among other things, imply that the phase speed is independent of the radial location and of the flow quantity considered, neither of which is true (cf. Crighton & Gaster 1976, and references there to relevant experimental work). This point is emphasized, on the theoretical side, by figure 11, which gives, for a fixed radial location and Strouhal number, the calculated variation of phase speed with x/D for two different physical quantities and for spinning modes with $m = 1, 2$ as well as for the axisymmetric mode; and figure 12 gives the variation in phase speed with Strouhal number for the $m = 1$ mode, as judged from the axial velocity fluctuation on the nozzle lip line $r = R$.

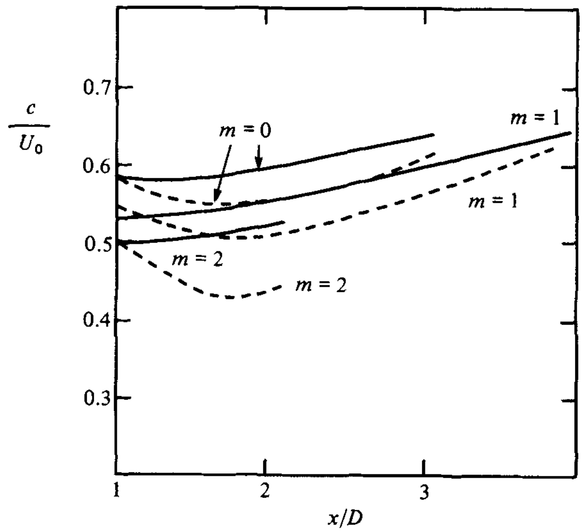

FIGURE 11. Axial variation in phase speed in shear layer ($r = R$). $St = 0.4$. Mode index $m = 0$, 1, 2, as indicated. ------, calculation for axial velocity fluctuation; ——, calculation for pressure fluctuation.

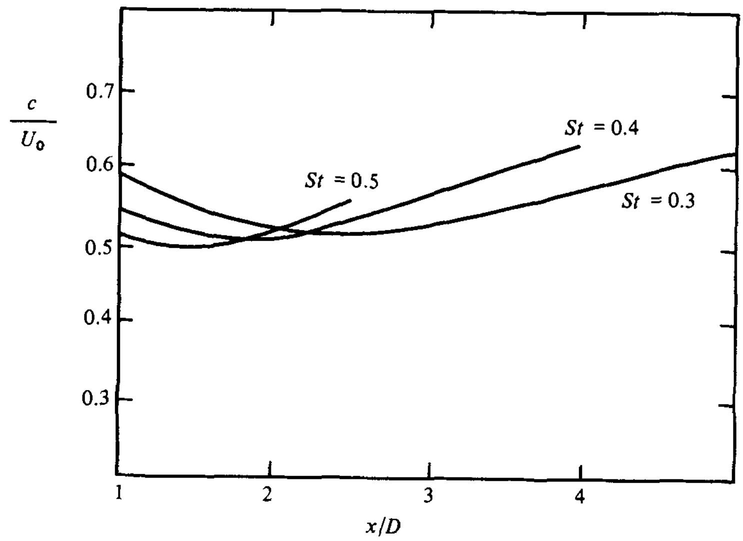

FIGURE 12. Axial variation in phase speed for shear-layer axial-velocity fluctuation ($r = R$). $St = 0.3$, 0.4, 0.5, as indicated. Mode index $m = 1$.

4. Discussion

A linear instability analysis of plane-wave and higher azimuthal order coherent wavelike disturbances on a slowly diverging mean flow has been used to predict the radial distributions, axial growth rates and phase speeds of the flow variables in a forced turbulent jet. The theory is limited to axial locations upstream of that at which the local parallel flow sustains a neutral wave, and the bounds of validity of the theory have been stretched in the comparison with the nonlinear instability wave response of the experiments. In the light of this, the measured radial structure of the fluctuations is predicted surprisingly well. By simply matching the amplitude of one velocity component to the corresponding experimental value at a single radial location we find favourable agreement between theory and experiment for the whole radial structure of the axial and radial velocities and of the pressure. For the $m = 1$ mode, which is not defined by a single scalar stream function, this is a particularly demanding test of the theory; and previous comparisons of the theory of §2 in Crighton & Gaster (1976) and Plaschko (1979) have not involved any such discussion of the transverse eigenfunction structure.

These remarks lend considerable support to the widely used, but hitherto largely unjustified, use of 'shape assumptions' in nonlinear stability theory (Stuart 1963; Ko, Kubota & Lees 1970). Typically, a shape assumption asserts that the transverse distributions of velocity and pressure will be calculated from linear parallel-flow instability theory at each axial station, and that these distributions will subsequently be used in some form of energy equation which will then govern the nonlinear streamwise evolution of the disturbance amplitudes. Several applications of this type of method to the large-scale orderly jet structure problem have recently been made (Chan 1977; Mankbadi & Liu 1981), but there the evidence presented for the shape assumption was extremely scant, and restricted in the first to pressure fluctuations and in the second to the axial velocity component only. Substantially more evidence in its favour than can be presented in this paper is given by Strange (1981).

The disparity between the linearity of the theory and the nonlinearity of the experiment precludes any assessment of the predicted axial growth. Nevertheless, a notable feature of the axial behaviour is the *decay* predicted by the multiple-scale analysis. In order to study decaying waves in the parallel-flow situation, it is necessary to bring in viscous forces, or equivalently, in the high-Reynolds-number limit, to deform the path for the radial integration of the inviscid equations into a complex plane and below the complex critical point, in the manner described, for example, by Gotoh (1968). An alternative method for proceeding axially beyond the location of maximum gain appears from the present work to be provided by introducing non-parallel effects, and in the present jet problem we suggest that this alternative is much more appropriate. We observe that the possibility, mentioned earlier as suggested by the work of Batchelor & Gill (1962), of continued growth of a spiral mode on the far-downstream jet, cannot be realized unless there is some direct excitation of the jet in that region. It appears that non-parallel effects will cause a freely developing disturbance there to *decay*, even though the analysis of Batchelor & Gill predicts the profile to remain locally unstable to a spinning mode with $m = 1$.

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